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Existence and Stability for Mathematical Models of Sedimentation–Consolidation Processes in Several Space Dimensions

R. Bürger

Institute of Mathematics A, University of Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart, Germany
E-mail: buerger@mathematik.uni-stuttgart.de

C. Liu

Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802
E-mail: liu@math.psu.edu

and

W. L. Wendland

Institute of Mathematics A, University of Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart, Germany

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In this paper, we study several spatially multidimensional initial-boundary value problems modelling sedimentation–consolidation processes of a flocculated suspension. This solid–fluid mixture is considered as two superimposed continuous media differing in density and viscosity. The phenomenological foundation and derivation of the mathematical model are based on the previous work by R. Bürger et al. (2000, *Z. Angew. Math. Mech*. **80**, 79–92). We study the full coupling of the conservation of mass equation and the conservation of linear momentum equation. For different types of regularization, we establish energy estimates. The dissipative nature of the whole system assures the existence as well as the stability (long time asymptotics) of a solution of the system (provided that the viscosities of the fluid are large enough). Moreover, the energy estimates might serve as the foundation for the design of numerical algorithms to simulate the system. © 2001 Elsevier Science

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1. INTRODUCTION

Mathematical models describing the sedimentation and consolidation of solid–liquid suspensions under the influence of gravity are of great importance for a variety of applications such as wastewater treatment, mineral processing, chemical and civil engineering, and manufacturing in the ceramical industry. The vast majority of equations proposed and analyzed for these processes in the engineering and mathematical literature are limited to one space dimension and are based on the kinematic sedimentation theory originating in the celebrated paper by Kynch [18] for ideal suspensions and its extensions such as continuous thickening in settler–clarifier units or, more recently, to flocculated suspensions forming compressible sediments. The former case gives rise to scalar conservation laws with singular source terms and a discontinuous flux function, while the latter leads to a scalar strongly degenerate parabolic–hyperbolic convection–diffusion equation.

While one-dimensional sedimentation models have been proposed, analyzed, and validated by numerous authors (see [8] for a comprehensive summary of this research), only a few groups of research workers have attempted to investigate spatially multidimensional sedimentation–consolidation equations. This is in part due to the fact that the solid–liquid separation takes place in one preferred direction, namely in that of the body force applied in order to exploit the density difference between the solid and the fluid, and that vessels are designed in such a way that flow transversal to this direction is minimized. On the other hand, solid–liquid separation vessels have features such as conical bottoms and feed and discharge openings that make a multidimensional treatment necessary.

From a mathematical point of view, the main difficulty in extending one of the well-known one-dimensional sedimentation–consolidation models to several space dimensions lies in the fact that not only the convection–diffusion equation describing the evolution of the solids concentration distribution has to be solved as in one space dimension, but that also additional equations describing the motion of the mixture have to be considered. These equations are usually, and should be, strongly coupled with the convection–diffusion equation, since the local concentration distribution affects the flow properties of the mixture.

To elucidate the importance of this coupling, we mention that there have been a few attempts to formulate spatially multidimensional sedimentation theories having in common that the degree of coupling was very weak [10, 12, 24, 25]. Most notably, the kinematic wave theory advanced in [24, 25], which can be viewed as a possible extension of Kynch's kinematic sedimentation theory [18] to two space dimensions, utilizes a very simple set of equations of motion for the mixture, which require that this coupling be described by boundary conditions. The resulting set of sedimentation equations, which can be easily extended to accommodate compression effects caused by flocculated suspensions [4], essentially implies that the concentration distribution is still one-dimensional (varying with height and time only), whereas the concentration profiles are imbedded in a two- or three-dimensional flow field in vessels of more general, e.g., conical, type. However, it turns out that these equations can neither be used for small times, since they are not stable with respect to perturbations of small initial data, nor do they lead to a stationary quiescent flow field for $t \to \infty$ since they predict circulatory clear liquid flow mechanisms. Bürger and Kunik [4] speculated that introducing the effect of mixture viscosity would provide for the necessary coupling, and damping of these flow mechanisms would lead to a more realistic model. The estimates established in this paper heavily depend on the concept of mixture viscosity and lend support to this view.

The starting point of the present paper is the sedimentation–consolidation model developed by Bürger et al. in [7], which is briefly recalled in Section 2. This model a priori provides enough terms for the coupling between the evolution of the concentration distribution, described by a strongly degenerate parabolic–hyperbolic convection–diffusion equation, and the motion of the mixture governed by a particular variant of the Navier–Stokes equations, on the entire computational domain. The authors' interest has been focused so far on the analysis and discretization of the strongly degenerate convection–diffusion equation, to which the model reduces in one space dimension; see [2, 3, 5, 6]. In particular, this equation and its discontinuous solutions were imbedded in the BV entropy solution framework going back to Kružkov [17]. We would like to mention that the ideas of this paper are not necessarily limited to the sedimentation–consolidation equations introduced in our paper [7] but can also possibly be used to analyze similar, independently developed multidimensional sedimentation–consolidation equations by Gustavsson and Oppelstrup [14, 15].

In this paper, a different approach, namely the energy approach, shall be employed to analyze the mentioned coupled system as a whole. Such an approach utilizes the nature of the dissipation of the volume average flow velocity. In other words, although discontinuities inevitably appear in the concentration field ϕ, the velocity field will always be regular (smooth). Moreover, the velocity field will eventually stabilize the whole system.

Here, we explore two types of regularization, the parabolic one and the so-called higher order one. We include in the parabolic case a particular solid–fluid interaction term coming from the variational procedure. As discussed in [13, 21], this term is the source of the surface tension between the interfaces. However, in our treatment, such regularization requires that the Kynch batch flux density function $f(\phi)$ is "small" and that the diffusion function $a(\phi)$ is smooth. However, the energy estimates that we have for such systems are independent of the regularization process. This is an important step to study the limiting process. Of course, more elaborate techniques of viscosity methods will have to be applied in order to understand the limiting system completely. In the higher order regularization, the constraints on $f(\phi)$ and $a(\phi)$ are relaxed. However, these estimates will depend on the regularization parameter ε.

In this paper, we are mainly concerned with the full coupling of the (linear) momentum equation and the parabolic–hyperbolic conservation of mass equation, and we demonstrate the dissipative effect of the (fluid) viscosity. We point out that all the physical phenomena are retained in our approach. These results can be combined with the techniques and results from our previous work [7] in order to understand the physical and mathematical features of these models of sedimentation–consolidation processes.

2. MATHEMATICAL MODEL

We consider the system of equations proposed in [7] as a model of sedimentation–consolidation processes in closed vessels,

$$\partial_t \phi + \mathbf{q} \cdot \nabla \phi + \nabla \cdot (f(\phi)\mathbf{k}) = \Delta A(\phi), \tag{1}$$

$$\nabla \cdot \mathbf{q} = 0, \tag{2}$$

$$\begin{aligned} &-\rho(\phi)(\partial_t \mathbf{q} + (\mathbf{q}\cdot\nabla)\mathbf{q}) + \nabla\cdot(\mu(\phi)\mathbf{D}) \\ &\quad = \nabla p_{\mathrm{e}} + \nabla\sigma_{\mathrm{e}}(\phi) + \delta_\rho g \phi \mathbf{k} \\ &\qquad + \rho(\phi)\big[(\mathbf{q}\cdot\nabla)\mathbf{r}(\phi,\nabla\phi) + (\mathbf{r}(\phi,\nabla\phi)\cdot\nabla)\,\mathbf{q}\big] \\ &\qquad + \tilde{\mathbf{g}}(D_{\mathbf{x}}^{\alpha}\phi). \end{aligned} \tag{3}$$

Here ϕ is the sought volumetric solids concentration, t is time, $\mathbf{q}$ is the volumetric average velocity of the mixture, f is the Kynch batch flux density function [18], and $\mathbf{k}$ is the upwards pointing unit vector. The function A is defined as

$$A(\phi) := \int_0^{\phi} a(s)ds, \qquad a(\phi) := -\frac{f(\phi)\sigma_{\mathrm{e}}'(\phi)}{\delta_\rho g \phi}, \tag{4}$$

where $\delta_\rho > 0$ is the solid–fluid density difference, g is the acceleration of gravity, and σ_{e}' is the derivative of the effective solids stress function. Furthermore, constitutively, we set here

$$\rho(\phi) = \phi\rho_{\mathrm{s}} + (1-\phi)\rho_{\mathrm{f}}, \qquad \mu(\phi) = \phi\mu_{\mathrm{s}} + (1-\phi)\mu_{\mathrm{f}} \tag{5}$$

which define the local density and viscosity of the mixture, respectively. Here ρ_s and ρ_f are the (constant) mass densities and μ_s and μ_f are the dynamic phase viscosities of the solid and the fluid. The functions ρ and μ are the mixture density and viscosity function (which can be assumed as the effective density and viscosity). We set

$$\mathbf{D} := \frac{1}{2}\left[\nabla \mathbf{q} + (\nabla \mathbf{q})^{\mathrm{T}}\right].$$

p_e denotes the excess pore pressure,

$$\mathbf{r}(\phi, \nabla\phi) := \frac{f(\phi)}{\rho(\phi)g}(\nabla\sigma_e(\phi) + \delta_\rho g\phi\mathbf{k}),$$

and $\tilde{\mathbf{g}}$ is the function of ϕ and its partial derivatives with respect to the spatial variables of up to second order,

$$\begin{aligned}\tilde{\mathbf{g}} = \nabla\cdot\bigg(\mu(\phi)\bigg[\nabla\phi\mathbf{v}_r(\phi,\nabla\phi) + (\nabla\phi\mathbf{v}_r(\phi,\nabla\phi))^{\mathrm{T}} - \frac{2}{3}(\nabla\phi\cdot\mathbf{v}_r(\phi,\nabla\phi))\mathbf{I}\bigg]\bigg)\\ + \rho(\phi)\delta_\rho \mathrm{D}_t\bigg(\frac{\phi(1-\phi)}{\rho(\phi)}\mathbf{v}_r(\phi,\nabla\phi)\bigg)\\ + \rho_s\rho_f\nabla\cdot\bigg(\frac{\phi(1-\phi)}{\rho(\phi)}\mathbf{v}_r(\phi,\nabla\phi)\mathbf{v}_r(\phi,\nabla\phi)\bigg),\end{aligned} \tag{6}$$

where the function $\mathbf{v}_r$, the so-called solid–fluid drift or relative velocity, is given by

$$\mathbf{v}_r(\phi, \nabla\phi) = \frac{\delta_\rho\phi(1-\phi)}{\rho(\phi)}\mathbf{r}(\phi).$$

The fact that the function $\tilde{\mathbf{g}}$ does not depend on third-order derivatives of ϕ is due to the particular composition (5) of the mixture viscosity from the phase viscosities; see [7] for details.

The suspension is described here by the three concentration-dependent (empirical) functions f, σ_e, and μ, which have to be determined by experiments. These functions are assumed to satisfy the generic assumptions

$$f \in C^1[0, \phi_{\max}], \quad f(\phi)\begin{cases} < 0 & \text{for } 0 < \phi < \phi_{\max}, \\ = 0 & \text{otherwise,}\end{cases} \tag{7}$$

where $\phi_{\max}$ is the maximum solids concentration,

$$\sigma_e'(\phi) \equiv \frac{d\sigma_e(\phi)}{d\phi}\begin{cases} = 0 & \text{for } \phi \le \phi_c, \\ > 0 & \text{for } \phi > \phi_c,\end{cases} \tag{8}$$

where ϕ_c with $0 \le \phi_c \le \phi_{\max}$ is the critical concentration at which the solid flocs touch each other, and

$$\mu(\phi) > \mu_{\min} > 0 \qquad \text{for } 0 \le \phi \le \phi_{\max}, \tag{9}$$

where $\mu_{\min}$ denotes a minimum viscosity, for example that of the pure fluid.

Typical examples of these functions used in solid–liquid separation models include

$$f(\phi) = u_\infty \phi(1 - \phi/\phi_{\max})^C, \qquad C > 1,$$

according to [22], where $u_\infty < 0$ denotes the settling velocity of a single particle in pure fluid,

$$\sigma_e(\phi) = \begin{cases} 0 & \text{for } \phi \le \phi_c, \\ \sigma_0((\phi/\phi_c)^n - 1) & \text{for } \phi > \phi_c, \end{cases} \qquad n > 1,$$

and the effective viscosity function [26]

$$\mu(\phi) = \mu_0(1 - \phi/\phi_{\max})^{-2.5\phi_{\max}},$$

where μ_0 denotes the viscosity of the pure fluid; i.e., we have $\mu_{\min} = \mu_0$.

It is emphasized that since $\mathbf{r}$ and $\mathbf{v}_r$ are mere shorthand notations, the unique velocity occurring in Eqs. (1)–(3) is the volume average mixture velocity $\mathbf{q}$. In fact, the quantities to be determined from (1)–(3) are the velocity field $\mathbf{q}$, the volumetric solids concentration ϕ, and the excess pore pressure p_e.

Due to the assumed generic properties for the functions f and σ_e, the function $a(\phi)$, denoting the diffusion coefficient of Eq. (1) since $\Delta A(\phi) = \nabla \cdot (a(\phi)\nabla\phi)$, satisfies

$$a(\phi) \begin{cases} = 0 & \text{for } \phi \le \phi_c \text{ and } \phi = \phi_{\max}\text{; i.e., Eq. (1) is hyperbolic,} \\ > 0 & \text{for } \phi_c < \phi < \phi_{\max}\text{; i.e., Eq. (1) is parabolic.} \end{cases}$$

For the system of equations (1)–(3), we consider the alternative sets of boundary conditions

$$\mathbf{q}(\mathbf{x}, t) = \mathbf{q}_0(\mathbf{x}), \quad \phi(\mathbf{x}, t) = \phi_0(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \partial\Omega, \quad t > 0 \tag{10}$$

or

$$\mathbf{q}(\mathbf{x}, t) = \mathbf{q}_0(\mathbf{x}), \quad \partial_{\mathbf{n}}\phi(\mathbf{x}, t) = g(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \partial\Omega, \quad t > 0, \tag{11}$$

where $\mathbf{n}$ is the outwards pointing normal vector defined on $\partial\Omega$. Suitable initial conditions are

$$\mathbf{q}(\mathbf{x}, 0) = \mathbf{q}_0(\mathbf{x}), \quad \phi(\mathbf{x}, 0) = \phi_0(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \Omega. \tag{12}$$

Note that the boundary datum for ϕ in (10) is given as the trace for $\mathbf{x} \to \partial\Omega, \mathbf{x} \in \Omega$ of the initial function ϕ_0. Other mixed type (oblique) boundary conditions could also be considered in a similar manner.

An essential property of (1)–(3) is the boundedness of the solution ϕ, which follows from the standard maximum principle [20]:

LEMMA 1. *Assume that* $(\phi, \mathbf{q}, p_e)$ *is a solution of* (1)–(3). *If the initial and boundary conditions are chosen such that* $|\phi| \le \alpha < \infty$ *on* $\Omega \times \{0\} \cup \partial\Omega \times (0, T]$, *then* $|\phi| \le \alpha$ a.e. *on* $\Omega \times (0, T]$ *for all* $T > 0$.

3. ENERGY ESTIMATES

3.1. *Parabolic Regularization*

Instead of the nonlinear system (1)–(3), we consider here the system of equations

$$\partial_t \phi + \mathbf{q} \cdot \nabla \phi + a_1(\mathbf{x}, t)(\mathbf{k} \cdot \nabla)\phi + b_1(\mathbf{x}, t)\phi = \nabla \cdot (a(\phi)\nabla\phi), \tag{13}$$

$$\nabla \cdot \mathbf{q} = 0, \tag{14}$$

$$\begin{aligned} \rho(\phi)(\partial_t \mathbf{q} + \mathbf{q} \cdot \nabla \mathbf{q}) - \nabla p_{\mathrm{e}} - \nabla \cdot (\mu(\phi)\mathrm{D}) \\ = \mathbf{a}_2 \phi + b_2 \nabla\phi + \mathbf{c}_2 \Delta\phi + e_2 \mathrm{D}_t \nabla \phi, \end{aligned} \tag{15}$$

where $a_1, b_1, c_1, \mathbf{c}_2$, and e_2 are given functions of $\mathbf{x}$ and t. The functions $\mathbf{a}_2, b_2$ are also allowed to depend linearly on $\mathbf{q}$ and $\nabla\mathbf{q}$. They can be regarded as the linearization of the Kynch batch flux density function and the effective solid stress function with respect to ϕ.

First of all, we replace (46) by the strictly parabolic equation, which coincides with the well-known regularization appearing in the viscosity method for hyperbolic [17, 23] and strongly degenerate parabolic equations [2, 5, 27],

$$\begin{aligned} \partial_t \phi^\varepsilon + (\mathbf{q}^\varepsilon \cdot \nabla)\phi^\varepsilon + a_1^\varepsilon(\mathbf{x}, t)(\mathbf{k} \cdot \nabla)\phi^\varepsilon + b_1^\varepsilon(\mathbf{x}, t)\phi^\varepsilon \\ = \nabla \cdot \big((a(\phi) + \varepsilon^2)\nabla\phi^\varepsilon\big), \end{aligned} \tag{16}$$

where $\varepsilon > 0$ is a small regularization parameter and a_1^ε is an appropriate regularization of the function a_1. Unless otherwise stated, the rest of the discussion in this section will exclusively refer to the regularized equation (16), and the parameter ε will be omitted in the notation. Also, in this section, we study the case that $a(\phi)$ is a bounded, smooth function. The case $a(\phi)$ has a jump will be treated in the next section involving the so-called higher order regularization.

We are interested in the problem with the boundary condition

$$\mathbf{q}(\mathbf{x}, t) = 0, \quad \phi(\mathbf{x}, t) = \phi_0(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \partial\Omega, \quad t > 0, \tag{17}$$

and the initial condition

$$\mathbf{q}(\mathbf{x}, 0) = \mathbf{q}_0(\mathbf{x}), \quad \phi(\mathbf{x}, 0) = \phi_0(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \Omega. \tag{18}$$

Note that also here the boundary datum for ϕ in (17) is the boundary trace of the initial function ϕ_0. In the same manner, we could also consider boundary conditions of Neumann type,

$$\mathbf{q}(\mathbf{x}, t) = 0, \quad \partial_{\mathbf{n}}\phi(\mathbf{x}, t) = 0 \qquad \text{for all } \mathbf{x} \in \partial\Omega, \quad t > 0.$$

We point out that for other more general boundary conditions, we can solve the stationary problem associated with the system (1)–(3) and subtract its solution from the system. This will result in the system with an extra external force term, which satisfies the initial and boundary conditions (17) and (18).

In order to derive an energy estimate, we first calculate that

$$\begin{aligned}\frac{d}{dt}\int_\Omega\left(\frac{1}{2}\rho(\phi)|\mathbf{q}|^2+\frac{a(\phi)+\varepsilon^2}{2}|\nabla\phi|^2\right)d\mathbf{x}\\ =\int_\Omega\bigg(\frac{1}{2}\rho'(\phi)\partial_t\phi|\mathbf{q}|^2+\rho(\phi)\mathbf{q}\cdot\partial_t\mathbf{q}\\ +(a(\phi)+\varepsilon^2)\nabla\phi\cdot\partial_t(\nabla\phi)+\frac{1}{2}a'(\phi)|\nabla\phi|^2\partial_t\phi\bigg)\,d\mathbf{x}\end{aligned}$$

and then integrate by parts the third term. By using (13) and (15), together with the boundary condition (17) and the fact that $\partial_t\phi=0$ on the boundary, we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega\left(\frac{1}{2}\rho(\phi)|\mathbf{q}|^2+\frac{a(\phi)+\varepsilon^2}{2}|\nabla\phi|^2\right)d\mathbf{x}\\ =\int_\Omega\bigg(\frac{1}{2}\rho'(\phi)\partial_t\phi|\mathbf{q}|^2+\mathbf{q}\cdot\Big(-\rho(\phi)(\mathbf{q}\cdot\nabla)\mathbf{q}+\nabla p_{\mathrm{e}}\\ +\nabla\cdot(\mu(\phi)\mathrm{D})+\mathbf{a}_2\phi+b_2\nabla\phi+\mathbf{c}_2\Delta\phi+e_2\mathrm{D}_t\nabla\phi\Big)\\ +\partial_t\phi(-\partial_t\phi-(\mathbf{q}\cdot\nabla)\phi-a_1(\mathbf{k}\cdot\nabla)\phi-b_1\phi)\\ +\frac{1}{2}a'(\phi)|\nabla\phi|^2\partial_t\phi\bigg)\,d\mathbf{x}\\ =\int_\Omega\Big(-\mu(\phi)|\nabla\mathbf{q}|^2-|\partial_t\phi|^2\Big)d\mathbf{x}\\ +\int_\Omega\bigg(\frac{1}{2}\rho'(\phi)\partial_t\phi|\mathbf{q}|^2-(\rho(\phi)(\mathbf{q}\cdot\nabla)\mathbf{q})\cdot\mathbf{q}+\mathbf{a}_2\phi\mathbf{q}+b_2\nabla\phi\mathbf{q}\\ -\mathbf{c}_2\nabla\phi\nabla\mathbf{q}-e_2\mathrm{D}_t\phi\nabla\mathbf{q}-((\mathbf{q}\cdot\nabla)\phi\partial_t\phi+a_1(\mathbf{k}\cdot\nabla)\phi\partial_t\phi\\ +b_1\phi\partial_t\phi+c_1\partial_t\phi)+\frac{1}{2}a'(\phi)|\nabla\phi|^2\partial_t\phi\bigg)d\mathbf{x}.\end{aligned}\tag{19}$$

Equation (19) is normally referred to as an energy estimate, since the first term $\frac{1}{2}\rho(\phi)|\mathbf{q}|^2$ is the kinetic energy of the fluid and the second term $[(a(\phi)+\varepsilon^2)/2]|\nabla\phi|^2$ represents the internal energy due to the uneven distribution of the solid particles. The control of this term is equivalent to the second law of thermodynamics in the isothermal situation; that is, the temperature remains the same.

The following interpolating inequality is due to Ladyzhenskaya [19]:

LEMMA 2. *For $u \in H_0^1(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, the following inequality is valid*:

$$\|u\|_{L^4(\Omega)} \leq \begin{cases} C\|u\|_{L^2(\Omega)}^{1/2}\|\nabla u\|_{L^2(\Omega)}^{1/2} & \text{if } n = 2, \\ C\|u\|_{L^2(\Omega)}^{1/4}\|\nabla u\|_{L^2(\Omega)}^{3/4} & \text{if } n = 3. \end{cases} \tag{20}$$

Using (20) we can now estimate the terms appearing on the right-hand side of (19) as follows. We will demonstrate the process in the two-dimensional case. The three-dimensional case is treated in exactly the same way.

For any chosen $\delta > 0$, the following inequality is valid:

$$\begin{aligned} \int_\Omega \frac{1}{2}\rho'(\phi)\partial_t\phi|\mathbf{q}|^2\, d\mathbf{x} &\leq C\|\partial_t\phi\|_{L^2}\|\mathbf{q}\|_{L^4}^2 \\ &\leq C\|\partial_t\phi\|_{L^2}\|\nabla\mathbf{q}\|_{L^2}\|\mathbf{q}\|_{L^2} \\ &\leq \delta\|\partial_t\phi\|_{L^2}^2 + C(\delta)\|\mathbf{q}\|_{L^2}^2\|\nabla\mathbf{q}\|_{L^2}^2. \end{aligned} \tag{21}$$

Moreover, we have

$$\int_\Omega(\rho(\phi)(\mathbf{q}\cdot\nabla)\mathbf{q})\cdot\mathbf{q}\, d\mathbf{x} \leq C\|\nabla\mathbf{q}\|_{L^2}\|\mathbf{q}\|_{L^4}^2 \leq C\|\nabla\mathbf{q}\|_{L^2}^2\|\mathbf{q}\|_{L^2}, \tag{22}$$

$$2\int_\Omega \phi|\mathbf{q}|\, d\mathbf{x} \leq \|\phi\|_{L^2}\|\mathbf{q}\|_{L^2}, \tag{23}$$

$$2\int_\Omega \nabla\phi\cdot\mathbf{q}\, d\mathbf{x} \leq \|\nabla\phi\|_{L^2}\|\mathbf{q}\|_{L^2}, \tag{24}$$

$$\int_\Omega |\mathrm{D}_t\phi|\nabla\mathbf{q}\, d\mathbf{x} \leq \delta\|\mathrm{D}_t\phi\|_{L^2}^2 + C(\delta)\|\nabla\mathbf{q}\|_{L^2}^2, \tag{25}$$

$$\int_\Omega \phi\partial_t\phi\, d\mathbf{x} \leq \|\phi\|_{L^2}\|\partial_t\phi\|_{L^2}. \tag{26}$$

The following three terms come from the occurrence of the functions $\mathbf{a}_2$ and b_2 in the momentum equation (15). Since they can linearly depend on $\mathbf{q}$ and $\nabla\mathbf{q}$, it is sufficient to use the estimates

$$\int_\Omega |\mathbf{q}||\phi||\mathbf{q}|\, d\mathbf{x} \leq C\|\mathbf{q}\|_{L^2}\|\mathbf{q}\|_{L^4}\|\phi\|_{L^4}, \tag{27}$$

$$\int_\Omega |\mathbf{q}||\nabla\phi||\mathbf{q}|\, d\mathbf{x} \leq C\|\mathbf{q}\|_{L^4}\|\mathbf{q}\|_{L^4}\|\nabla\phi\|_{L^2}, \tag{28}$$

$$\int_\Omega |\nabla\mathbf{q}||\phi||\mathbf{q}|\, d\mathbf{x} \leq C\|\nabla\mathbf{q}\|_{L^2}\|\mathbf{q}\|_{L^4}\|\phi\|_{L^4} \tag{29}$$

as well as

$$
\begin{aligned}
&\int_\Omega |\nabla \mathbf{q}||\nabla\phi||\mathbf{q}|\,d\mathbf{x}\\
&\quad \le C\|\nabla\mathbf{q}\|_{L^2}\|\nabla\phi\|_{L^4}\|\mathbf{q}\|_{L^4}\\
&\quad \le C\|\nabla\mathbf{q}\|_{L^2}\Big(\|\nabla\phi\|_{L^2}+\|\nabla\phi\|_{L^2}^{1/2}\|\Delta\phi\|_{L^2}^{1/2}\Big)\|\mathbf{q}\|_{L^2}^{1/2}\|\nabla\mathbf{q}\|_{L^2}^{1/2}\\
&\quad \le C\|\nabla\mathbf{q}\|_{L^2}\|\nabla\phi\|_{L^2}\|\mathbf{q}\|_{L^2}^{1/2}\|\nabla\mathbf{q}\|_{L^2}^{1/2}+C(\varepsilon)\|\nabla\mathbf{q}\|_{L^2}\|\nabla\phi\|_{L^2}^{1/2}\\
&\qquad \times\Big(\|\partial_t\phi\|_{L^2}+\|\mathbf{q}\|_{L^2}^{1/2}\|\nabla\mathbf{q}\|_{L^2}^{1/2}\|\nabla\phi\|_{L^2}+\|\mathbf{q}\|_{L^2}\|\nabla\mathbf{q}\|_{L^2}\\
&\qquad\qquad +\|a_1\|_{L^\infty}\|\nabla\phi\|_{L^2}+\|b_1\|_{L^\infty}\|\phi\|_{L^2}\\
&\qquad\qquad +\|a'(\phi)\|_{L^\infty}\|\nabla\phi\|_{L^2}^2\Big)^{1/2}\|\mathbf{q}\|_{L^2}^{1/2}\|\nabla\mathbf{q}\|_{L^2}^{1/2}\\
&\quad \le \delta\|\partial_t\phi\|_{L^2}^2+C(\delta,\varepsilon)\Big(\|\nabla\phi\|_{L^2}^2\|\mathbf{q}\|_{L^2}+\|\nabla\phi\|_{L^2}^{3/2}\|\mathbf{q}\|_{L^2}^{3/2}\\
&\qquad\qquad +\|\nabla\phi\|_{L^2}\|\mathbf{q}\|_{L^2}+\|\phi\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2\Big)\|\nabla\mathbf{q}\|_{L^2}^2. \qquad (30)
\end{aligned}
$$

The last estimate is due to the parabolic regularity of the equation (16), together with Lemma 1,

$$
\begin{aligned}
\|\Delta\phi\|_{L^2}\le C(\varepsilon)\Big(&\|\partial_t\phi\|_{L^2}+\|\mathbf{q}\|_{L^2}^{1/2}\|\nabla\mathbf{q}\|_{L^2}^{1/2}\|\nabla\phi\|_{L^4}+\|a'\|_{L^\infty}\|\nabla\phi\|_{L^4}^2\\
&+\|a_1\|_{L^\infty}\|\nabla\phi\|_{L^2}+\|b_1\|_{L^\infty}\|\phi\|_{L^2}\Big), \qquad (31)
\end{aligned}
$$

and by Cauchy's inequality and Sobolev's inequality, we have

$$
\|\nabla\phi\|_{L^4}\le C\|\nabla\phi\|_{L^2}^{1/2}\Big(\|\nabla\phi\|_{L^2}^{1/2}+\|\Delta\phi\|_{L^2}^{1/2}\Big). \qquad (32)
$$

Hence, by using Cauchy's inequality again, we obtain

$$
\begin{aligned}
\|\mathrm{D}_{\mathbf{x}}^2\phi\|_{L^2}\le C(\varepsilon)\Big(&\|\partial_t\phi\|_{L^2}+\|\mathbf{q}\|_{L^2}^{1/2}\|\nabla\mathbf{q}\|_{L^2}^{1/2}\|\nabla\phi\|_{L^2}+|\mathbf{q}\|_{L^2}\|\nabla\mathbf{q}\|_{L^2}\|\nabla\phi\|_{L^2}\\
&+\|a_1\|_{L^\infty}\|\nabla\phi\|_{L^2}+\|b_1\|_{L^\infty}\|\phi\|_{L^2}+\|a'\|_{L^\infty}\|\nabla\phi\|_{L^2}^2\\
&+\|a'\|_{L^\infty}\|\nabla\phi\|_{L^2}\|\Delta\phi\|_{L^2}\Big), \qquad (33)
\end{aligned}
$$

where $\mathrm{D}_{\mathbf{x}}^2\phi$ denotes the Hessian of ϕ, i.e., the matrix containing all second-order derivatives $\partial^2_{x_ix_j}\phi$, $i, j = 1, \ldots, d$. In case that $\|a'(\phi)\|_{L^\infty}\|\nabla\phi\|_{L^2}$ is small enough, we find

$$\begin{aligned}\|\mathrm{D}_{\mathbf{x}}^2\phi\|_{L^2} \le C(\varepsilon)\Big(&\|\partial_t\phi\|_{L^2} + \|\mathbf{q}\|_{L^2}^{1/2}\|\nabla\mathbf{q}\|_{L^2}^{1/2}\|\nabla\phi\|_{L^2} \\ &+ \|\mathbf{q}\|_{L^2}\|\nabla\mathbf{q}\|_{L^2}\|\nabla\phi\|_{L^2} + \|a_1\|_{L^\infty}\|\nabla\phi\|_{L^2} \\ &+ \|b_1\|_{L^\infty}\|\phi\|_{L^2} + \|a'\|_{L^\infty}\|\nabla\phi\|_{L^2}^2\Big). \end{aligned} \tag{34}$$

The next term comes from the term describing the convective transport at the local velocity $\mathbf{q}$ of the concentration distribution in (13). Again, in the case that $\|a'(\phi)\|_{L^\infty}\|\nabla\phi\|_{L^2}$ is small enough, we have

$$\begin{aligned}&\int_\Omega (\mathbf{q}\cdot\nabla\phi)\partial_t\phi\,d\mathbf{x} \\ &\quad\le C\|\partial_t\phi\|_{L^2}\|\nabla\phi\|_{L^4}\|\mathbf{q}\|_{L^4} \\ &\quad\le C\|\partial_t\phi\|_{L^2}\Big(\|\nabla\phi\|_{L^2}+\|\nabla\phi\|_{L^2}^{1/2}\|\Delta\phi\|_{L^2}^{1/2}\Big)\|\mathbf{q}\|_{L^2}^{1/2}\|\nabla\mathbf{q}\|_{L^2}^{1/2} \\ &\quad\le C\|\partial_t\phi\|_{L^2}\|\nabla\phi\|_{L^2}\|\mathbf{q}\|_{L^2}^{1/2}\|\nabla\mathbf{q}\|_{L^2}^{1/2}+C(\varepsilon)\|\partial_t\phi\|_{L^2} \\ &\qquad\times\|\nabla\phi\|_{L^2}^{1/2}\Big(\|\partial_t\phi\|_{L^2}+\|\mathbf{q}\|_{L^2}^{1/2}\|\nabla\mathbf{q}\|_{L^2}^{1/2}\|\nabla\phi\|_{L^2} \\ &\qquad\qquad+\|\mathbf{q}\|_{L^2}\|\nabla\mathbf{q}\|_{L^2}\|a_1\|_{L^\infty}\|\nabla\phi\|_{L^2}+\|b_1\|_{L^\infty}\|\phi\|_{L^2} \\ &\qquad\qquad+\|a'(\phi)\|_{L^\infty}\|\nabla\phi\|_{L^2}^2\Big)^{1/2}\|\mathbf{q}\|_{L^2}^{1/2}\|\nabla\mathbf{q}\|_{L^2}^{1/2} \\ &\quad\le\delta\|\partial_t\phi\|_{L^2}^2+C(\delta,\varepsilon) \\ &\qquad\times\Big(\|\nabla\phi\|_{L^2}^2\|\mathbf{q}\|_{L^2}+\|\nabla\phi\|_{L^2}\|\mathbf{q}\|_{L^2}+\|\phi\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2\Big)\|\nabla\mathbf{q}\|_{L^2}^2. \end{aligned} \tag{35}$$

Here we again use the parabolic estimate (34) together with Lemma 1. Furthermore, estimate (34) also gives

$$\begin{aligned}&\int_\Omega \frac{1}{2}a'(\phi)|\nabla\phi|^2\partial_t\phi\,d\mathbf{x} \\ &\quad\le C\|a'\|_{L^\infty}\|\partial_t\phi\|_{L^2}\|\nabla\phi\|_{L^4}^2 \\ &\quad\le C\|a'\|_{L^\infty}\|\partial_t\phi\|_{L^2}\|\nabla\phi\|_{L^2}\|\Delta\phi\|_{L^2} \\ &\quad\le C\|a'\|_{L^\infty}\|\partial_t\phi\|_{L^2}\|\nabla\phi\|_{L^2} \\ &\qquad\times\Big(\|\partial_t\phi\|_{L^2}+\|\mathbf{q}\|_{L^2}\|\nabla\mathbf{q}\|_{L^2}\|\nabla\phi\|_{L^2}+\|\nabla\phi\|_{L^2}^{1/2}\Big) \\ &\quad\le C\|a'\|_{L^\infty}^2\|\nabla\phi\|_{L^2}\|\partial_t\phi\|_{L^2}^2+C\|\nabla\phi\|_{L^2}^{3/2}\|\partial_t\phi\|_{L^2} \\ &\qquad+C\|\nabla\phi\|_{L^2}^3\|\mathbf{q}\|_{L^2}^2\|\nabla\mathbf{q}\|_{L^2}^2, \end{aligned} \tag{36}$$

where we use that a is smooth and ϕ is bounded. Finally, we have

$$\int_\Omega a_1(\mathbf{k}\cdot\nabla)\phi\partial_t\phi\;d\mathbf{x}\le\|a_1\|_{L^\infty}\|\partial_t\phi\|_{L^2}\|\nabla\phi\|_{L^2}. \tag{37}$$

Combining all these estimates, if we choose δ small enough, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\left(\frac{1}{2}\rho(\phi)|\mathbf{q}|^2+\frac{a(\phi)+\varepsilon^2}{2}|\nabla\phi|^2\right)d\mathbf{x}\\
&\quad+\big(\big(\mu_{\min}-C\|\mathbf{q}\|_{L^2}-C(\delta)\big(\|\nabla\phi\|_{L^2}^3\|\mathbf{q}\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2\|\mathbf{q}\|_{L^2}^2+\|\nabla\phi\|_{L^2}\big)\big)\\
&\qquad\times\|\nabla\mathbf{q}\|_{L^2}-C(\|\nabla\phi\|_{L^2}+\|\phi\|_{L^2}+\|\mathbf{q}\|_{L^2})\big)\|\nabla\mathbf{q}\|_{L^2}\\
&\quad+\Big[\big(1-8\delta-C\|a'\|_{L^\infty}^2\|\nabla\phi\|_{L^2}\big)\|\partial_t\phi\|_{L^2}\\
&\qquad\quad-C\big(\|a_1\|_{L^\infty}\|\nabla\phi\|_{L^2}+\|b_1\|_{L^\infty}\|\phi\|_{L^2}\big)\Big]\|\partial_t\phi\|_{L^2}\le 0. \qquad (38)
\end{aligned}$$

Here $\mu_{\min}=\min\ \mu(\phi)>0$.

Remark 1. We notice that both functions a_1 and b_1 are derived from the Kynch batch flux density functions $f(\phi)$. The nonlinearity of this function gives rise to the formation of discontinuities (shocks) in the solution of the limiting system (13)–(15). In this section, we linearized such a function. Moreover, since the function $f(\phi)$ is equipped with the scaling reciprocal of the space variables (or the square root of the time), we assume that a_1 and b_1 are small, in a given bounded spatial and time region.

THEOREM 1. *Assume that* $(\mathbf{q},\phi)$ *is a* (*smooth*) *solution of the system* (14)–(18) *and* $\|a_1\|_{L^\infty}$ *and* $\|b_1\|_{L^\infty}$ *are small enough*, *and that*, *in addition*, *one of the following conditions is satisfied*:

(a) (*large viscosity* I) *There exist constants* K *and* ν_2, *depending on all other previously established constants and the initial datum such that* $\mu_{\min}\ge K$ *and* $\|\nabla\phi_0\|_{L^2}<\nu_2$.

(b) (*large viscosity* II) *There exist constants* K *and* k, *depending on all other previously established constants and the initial datum such that* $\mu_{\min}\ge K$ *and* $\|a'(\phi)\|_{L^\infty}\le k$.

(c) (*small initial datum*) *If* $\mu_{\min}$ *is big depending on all other previously established constants* (*but not the initial datum*), *we assume that there exists a constant* ν_1 *depending on all other previously established constants*, *such that*

$$\int_\Omega\left(\frac{1}{2}|\mathbf{q}_0|^2+\frac{1}{2}|\nabla\phi_0|^2\right)d\mathbf{x}\le\nu_1.$$

Then the following estimates hold:

$$\int_\Omega\left(\frac{1}{2}|\mathbf{q}|^2+\frac{1}{2}|\nabla\phi|^2\right)d\mathbf{x}(t)\le M(\varepsilon)\qquad \textit{for all } 0<t<\infty, \qquad (39)$$

$$\int_0^\infty\int_\Omega|\nabla\mathbf{q}|^2\,d\mathbf{x}\,dt\le M_1(\varepsilon), \qquad (40)$$

$$\int_0^\infty\int_\Omega|\partial_t\phi|^2\,d\mathbf{x}\,dt\le M_2(\varepsilon). \qquad (41)$$

Proof. In order to prove the theorem, we first use the Poincaré inequality

$$\|\mathbf{q}\|_{L^2} \leq C\|\nabla\mathbf{q}\|_{L^2}.$$

Second, we see from (34) that if $\|a_1\|_{L^\infty}$ and $\|b_1\|_{L^\infty}$ are small enough, we have that

$$\|\nabla\phi\|_{L^2} \leq C(\varepsilon)\big(\|\partial_t\phi\|_{L^2} + \|\mathbf{q}\|_{L^2}\|\nabla\mathbf{q}\|_{L^2}\|\nabla\phi\|_{L^2}\big).$$

We insert these into the inequality (38) and get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\left(\frac{1}{2}\rho(\phi)|\mathbf{q}|^2 + \frac{a(\phi)+\varepsilon^2}{2}|\nabla\phi|^2\right)d\mathbf{x}\\
&\quad+\big(\mu_{\min} - C\|\mathbf{q}\|_{L^2} - C(\delta)\big(\|\nabla\phi\|_{L^2}^3\|\mathbf{q}\|_{L^2}^2 + \|\nabla\phi\|_{L^2}^2\|\mathbf{q}\|_{L^2}^2\\
&\quad+\|\nabla\phi\|_{L^2}\big) - C\big)\|\nabla\mathbf{q}\|_{L^2}^2\\
&\quad+\big(1 - 10\delta - C\|a'\|_{L^\infty}^2\|\nabla\phi\|_{L^2} - C\|b_1\|_{L^\infty}\big)\|\partial_t\phi\|_{L^2}^2 \leq 0.
\end{aligned}\tag{42}$$

Now the conditions in all three cases guarantee that

$$\frac{d}{dt}\int_\Omega\left(\frac{1}{2}\rho(\phi)|\mathbf{q}|^2 + \frac{a(\phi)+\varepsilon^2}{2}|\nabla\phi|^2\right)d\mathbf{x} \leq 0 \tag{43}$$

and this yields the results.

We now want to prove case (c) differently by using (38) directly. We use the fact that $\mu_{\min} = \min\ \mu(\phi) > 0$. Let

$$\mathscr{A} := \int_\Omega\left(\frac{1}{2}|\mathbf{q}|^2 + \frac{1}{2}|\nabla\phi|^2\right)d\mathbf{x}, \qquad \mathscr{B} := \int_\Omega\left(\frac{1}{2}|\nabla\mathbf{q}_0|^2 + \frac{1}{2}|\partial_t\phi|^2\right)d\mathbf{x}$$

and assume that $\mathscr{A}$ is sufficiently small. Then, if $\mathscr{B}$ is large, the inequality

$$\frac{d\mathscr{A}}{dt} = \frac{d}{dt}\int_\Omega\left(\frac{1}{2}\rho(\phi)|\mathbf{q}|^2 + \frac{a(\phi)+\varepsilon^2}{2}|\nabla\phi|^2\right)d\mathbf{x} \leq 0 \tag{44}$$

holds again. On the other hand, if $\mathscr{B}$ is small, by using the same estimates as in the other cases, the term $\mathscr{A}$ will also be small. Let us illustrate this by the auxiliary inequality

$$\frac{d\mathscr{A}}{dt} + \big((\mu - \mathscr{A})\mathscr{B} - \mathscr{A}\big)\mathscr{B} \leq 0, \tag{45}$$

where $C\mathscr{A} \leq \mathscr{B}$. Assume we have $\mathscr{A}(t) < \mu$ at time t. If we can also show that

$$\mathscr{A}(t) \leq \frac{C\mu^2}{\mu+1},$$

then we have

$$\frac{\mathscr{A}}{C(\mu - \mathscr{A})} \leq \mu.$$

If $\mathcal{B} > \frac{\mathcal{A}}{\mu-\mathcal{A}}$, then from inequality (45), one gets $\frac{d\mathcal{A}}{dt} \leq 0$. If $\mathcal{B} < \frac{\mathcal{A}}{\mu-\mathcal{A}}$, then again

$$\mathcal{A} < \frac{\mathcal{A}}{C(\mu - \mathcal{A})} \leq \mu.$$

Hence we see that, if μ is big enough, $\mathcal{A}$ has a uniform bound

$$\mu = \min\left\{\frac{C\mu^2}{\mu+1}, \mu\right\}$$

in time t. The same argument can also be applied to the case (c) and this proves the theorem. ∎

We can see that the present estimates all depend on the regularization constant ε. All bounds will be lost as the constant approaches zero. So this estimate is too weak to be used for the viscosity method. Therefore, we propose next a slightly different system. Although this system is different from the original system, it is equipped with a variational formulation. This allows us to obtain the energy estimates independent of the regularization. Hence they will be compatible with the other viscosity methods.

3.2. *A Variational Formulation*

Instead of the nonlinear system (1)–(3), we consider in this section the system of equations

$$\partial_t\phi + \mathbf{q}\cdot\nabla\phi + a_1(\mathbf{x},t)(\mathbf{k}\cdot\nabla)\phi + b_1(\mathbf{x},t)\phi = \nabla\cdot(a(\phi)\nabla\phi) - \frac{1}{2}a'(\phi)|\nabla\phi|^2, \tag{46}$$

$$\nabla\cdot\mathbf{q} = 0, \tag{47}$$

$$\rho(\phi)(\partial_t\mathbf{q} + \mathbf{q}\cdot\nabla\mathbf{q}) - \nabla p_{\mathrm{e}} - \nabla\cdot(\mu(\phi)\mathbf{D}) = \mathbf{a}_2\phi + b_2\nabla\phi + \mathbf{c}_2\Delta\phi + e_2\partial_t\nabla\phi + \varepsilon^2\partial_t\nabla\phi, \tag{48}$$

where $a_1, b_1, c_1, \mathbf{c}_2$, and e_2 are given functions of $\mathbf{x}$ and t. The function $\mathbf{a}_2$ can also depend linearly on $\mathbf{q}$ and $\nabla\mathbf{q}$. This can be regarded as the linearization of the batch function and the stress function with respect to ϕ. We also require here the following additional constitutive "continuum" equation:

$$\partial_t\rho(\phi) + \mathbf{q}\cdot\nabla\rho(\phi) = 0. \tag{49}$$

Equation (49) expresses that we here consider ρ and ϕ as independent variables determined by separate dynamic laws. Consequently, the unknowns are ϕ, ρ, $\mathbf{q}$ and p_{e}. Thus ρ becomes an averaged effective quantity. This is consistent with the fact that $\rho\mathbf{q}$ is the linear momentum of the mixture and that $\mathbf{q}$ is an effective, averaged velocity.

We notice that some nonlinear terms remained in this system. The right-hand side of (46) now represents the variation of the internal energy

$\frac{1}{2}a(\phi)|\nabla\phi|^2$ with respect to ϕ. The second part of the expression models the interaction on the interface. The whole variation contains both, the diffusion and the reaction part of the equation. The last term $\varepsilon^2\partial_t\phi\nabla\phi$ in (48) can be viewed as the interaction of different particles. In fact, such a term can also be seen as one of the terms in the equation (6) for $\tilde{\mathbf{g}}$. Usually this term contributes for the surface tension on the interfaces.

In this section, we again study the case that $a(\phi)$ is a bounded, smooth function. The case that $a(\phi)$ has a jump will be treated in the next section with the so-called higher order regularization.

We are interested in the problem with the boundary conditions

$$\mathbf{q}(\mathbf{x},t)=0, \quad \phi(\mathbf{x},t)=\phi_0(\mathbf{x}) \qquad \text{for all } \mathbf{x}\in\partial\Omega, \quad t>0, \tag{50}$$

and the initial conditions

$$\mathbf{q}(\mathbf{x},0)=\mathbf{q}_0(\mathbf{x}), \quad \phi(\mathbf{x},0)=\phi_0(\mathbf{x}) \qquad \text{for all } \mathbf{x}\in\Omega. \tag{51}$$

Note that also here the boundary datum for ϕ in (50) is the boundary trace of the initial function ϕ_0. We also could consider boundary conditions of Neumann type; i.e.,

$$\mathbf{q}(\mathbf{x},t)=0, \quad \partial_{\mathbf{n}}\phi(\mathbf{x},t)=0 \qquad \text{for all } \mathbf{x}\in\partial\Omega, \quad t>0.$$

Again, we can solve the stationary problem associated with the system (1)–(3) and subtract the solution from the system which would result in the system with an extra external force term satisfying the initial and boundary conditions (50) and (51).

In order to derive an energy estimate, we first calculate that

$$\begin{aligned}\frac{d}{dt}\int_\Omega&\left(\frac{1}{2}\rho(\phi)|\mathbf{q}|^2+\frac{\varepsilon^2a(\phi)}{2}|\nabla\phi|^2\right)d\mathbf{x}\\&=\int_\Omega\left(\frac{1}{2}\rho'(\phi)\partial_t\phi|\mathbf{q}|^2+\rho(\phi)\mathbf{q}\cdot\partial_t\mathbf{q}\right.\\&\qquad\left.+\varepsilon^2a(\phi)\nabla\phi\cdot\partial_t(\nabla\phi)+\frac{\varepsilon^2}{2}a'(\phi)|\nabla\phi|^2\partial_t\phi\right)d\mathbf{x}\end{aligned}\tag{52}$$

and then integrate by parts the third term. By using (46) and (48), together with the boundary condition (50) and the fact that $\partial_t\phi=0$ on the boundary, we get

$$\begin{aligned}\frac{d}{dt}\int_\Omega&\left(\frac{1}{2}\rho(\phi)|\mathbf{q}|^2+\frac{\varepsilon^2a(\phi)}{2}|\nabla\phi|^2\right)d\mathbf{x}\\&=\int_\Omega\Big(\frac{1}{2}\rho'(\phi)\partial_t\phi|\mathbf{q}|^2+\mathbf{q}\cdot\Big(-\rho(\phi)(\mathbf{q}\cdot\nabla)\mathbf{q}+\nabla p_{\mathrm{e}}\\&\quad+\nabla\cdot(\mu(\phi)\mathbf{D})+\mathbf{a}_2\phi+b_2\nabla\phi+\mathbf{c}_2\Delta\phi+e_2\partial_t\nabla\phi+\varepsilon^2\phi_t\nabla\phi\Big)\\&\quad+\varepsilon^2\partial_t\phi\Big(-\partial_t\phi-(\mathbf{q}\cdot\nabla)\phi-a_1(\mathbf{k}\cdot\nabla)\phi-b_1\phi\\&\qquad\qquad-\frac{1}{2}a'(\phi)|\nabla\phi|^2\Big)+\frac{\varepsilon^2}{2}a'(\phi)|\nabla\phi|^2\partial_t\phi\Big)\,d\mathbf{x}.\end{aligned}\tag{53}$$

Now, several terms cancel out. These cancellations merely reflect the variational nature of the system. Now we apply integration by parts to the second term and use the continuum equation (49) to get

$$\frac{d}{dt}\int_{\Omega}\left(\frac{1}{2}\rho(\phi)|\mathbf{q}|^2+\frac{\varepsilon^2 a(\phi)}{2}|\nabla\phi|^2\right)d\mathbf{x}$$
$$=\int_{\Omega}(-\mu(\phi)|\nabla\mathbf{q}|^2-\varepsilon^2|\partial_t\phi|^2)\,d\mathbf{x}+\int_{\Omega}\big(\mathbf{a}_2\phi\mathbf{q}+b_2\nabla\phi\mathbf{q}$$
$$-\mathbf{c}_2\nabla\phi\nabla\mathbf{q}-e_2\partial_t\phi\nabla\mathbf{q}-\varepsilon^2(a_1(\mathbf{k}\cdot\nabla)\phi\partial_t\phi+b_1\phi\partial_t\phi)\big)\,d\mathbf{x}. \tag{54}$$

Here we can see that, although even we require that a be a smooth function, the energy estimate is independent of the derivative of a. Equation (54) is again an energy estimate, where the first term $\frac{1}{2}\rho(\phi)|\mathbf{q}|^2$ is again the kinetic energy and the second term $(\varepsilon^2 a(\phi)/2)|\nabla\phi|^2$ now represents the internal energy due to the uneven distribution of the solid particles.

Using inequality (20) from Lemma 2 we can estimate the terms appearing on the right-hand part of (54) in a similar manner as in Section 3.1. Using estimates (23) and (24), we see that the following inequalities are valid for any chosen $\delta > 0$:

$$\int_{\Omega}|\nabla\phi||\nabla\mathbf{q}|\,d\mathbf{x}\le\|\nabla\phi\|_{L^2}\|\nabla\mathbf{q}\|_{L^2}\le\delta\|\nabla\phi\|_{L^2}^2+C(\delta)\|\nabla\mathbf{q}\|_{L^2}^2, \tag{55}$$

$$\int_{\Omega}\partial_t\phi\nabla\mathbf{q}\;d\mathbf{x}\le\delta\|\partial_t\phi\|_{L^2}^2+C(\delta)\|\nabla\mathbf{q}\|_{L^2}^2. \tag{56}$$

Employing the estimates (26), (27), (29), (37), (55), and (56), we can establish the following theorem:

THEOREM 2. *Assume that* $(\mathbf{q},\phi)$ *is a (smooth) solution of the system* (47)–(51). *If there exist constants* K *depending on* e_2 *and* $\phi_{\max}$ *and* $\mu_{\min}\ge K$, *then for any* $0<T<\infty$, *we have*

$$\int_{\Omega}\left(\frac{1}{2}|\mathbf{q}|^2+\frac{1}{2}|\nabla\phi|^2\right)d\mathbf{x}(t)\le M(T)\qquad\text{for all } 0<t<T. \tag{57}$$

Moreover, the following estimates hold:

$$\int_0^T\int_{\Omega}|\nabla\mathbf{q}|^2\;d\mathbf{x}\;dt\le M_1(T), \tag{58}$$

$$\int_0^T\int_{\Omega}|\partial_t\phi|^2\;d\mathbf{x}\;dt\le M_2(T). \tag{59}$$

Proof. Combining all the estimates (22)–(29), (55), and (56), if we choose δ small enough, we have

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\rho(\phi)|\mathbf{q}|^2 + \frac{\varepsilon^2 a(\phi)}{2}|\nabla\phi|^2\right) d\mathbf{x}$$
$$+ \left(\mu_{\min} - e_2 C(\delta) - C\phi_{\max}\right)\|\nabla \mathbf{q}\|_{L^2}^2 + (\varepsilon^2 - 4\delta)\|\partial_t \phi\|_{L^2}$$
$$\leq C\int_\Omega \left(\frac{1}{2}\rho(\phi)|\mathbf{q}|^2 + \frac{\varepsilon^2 a(\phi)}{2}|\nabla\phi|^2\right) d\mathbf{x}. \tag{60}$$

Here $\mu_{\min} = \min\ \mu(\phi) > 0$. Under the conditions of the theorem, inequality (60) shows that

$$\int_\Omega \left(\frac{1}{2}\rho(\phi)|\mathbf{q}|^2 + \frac{\varepsilon^2 a(\phi)}{2}|\nabla\phi|^2\right) d\mathbf{x}$$

is bounded for any fixed time T. This implies the theorem. ∎

Remark 2. The local existence and the local energy estimate are true for all cases without any constraints on the initial condition and the viscosity.

The following lemma is a direct consequence of Theorem 2.

LEMMA 3. *Consider the regularized parabolic equation* (46) *and assume that $a(\phi) \geq \nu > 0$. Then for all sufficiently smooth initial and boundary data, and arbitrary, finite time T, the system admits a solution satisfying*

$$\mathbf{q} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \tag{61}$$

$$\sqrt{|a(\phi)|}\nabla\phi \in L^\infty(0, T; L^2(\Omega)), \tag{62}$$

$$\partial_t \phi \in L^2(0, T; L^2(\Omega)). \tag{63}$$

Moreover, the solution satisfies the estimates (39)–(41) *of Theorem* 1, *independent of the regularization.*

Remark 3. From the energy estimate (60) we can get the existence of the weak solution for the approximate system with any $\varepsilon > 0$ as follows: given $\tilde{\phi}$, we determine $\mathbf{q}$ and p_{e} from (2) and (3); inserting the solution $\mathbf{q}$ into (1) we get an updated solution ϕ. Since the fixed point of the map

$$L\colon \tilde{\phi} \mapsto \phi \tag{64}$$

is the solution of the original system, we may prove the existence of solutions by employing Schauder's fixed point theorem.

Begin with a function $\tilde{\phi}$ such that $\tilde{\phi} \in L^\infty(0, T; L^\infty(\Omega))$ and

$$\sqrt{|a(\tilde{\phi})|}\nabla\tilde{\phi} \in L^\infty(0, T; L^2(\Omega)), \qquad \partial_t\tilde{\phi} \in L^\infty(0, T; L^2(\Omega))$$

for any $0 < T < \infty$, with bounds

$$\left\| \sqrt{|a(\tilde{\phi})|}\nabla\tilde{\phi} \right\|_{L^\infty(0,T;L^2(\Omega))} + \|\tilde{\phi}\|_{L^\infty(0,T;L^\infty(\Omega))} + \|\partial_t\tilde{\phi}\|_{L^\infty(0,T;L^2(\Omega))} \le N.$$

Substitute this into the equations (2) and (3), and solve the Navier–Stokes type equations for $\mathbf{q}$ and p_e with

$$\mathbf{q} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$$

for any $\tilde{\phi}$ with the bounds

$$\|\mathbf{q}\|_{L^\infty(0,T;L^2(\Omega))} + \|\mathbf{q}\|_{L^2(0,T;H^1(\Omega))} \le N_1(N, T_1)$$

for some small T_1. Finally, we substitute this back into (1) to get

$$\phi \in L^\infty(0, T_1; L^\infty(\Omega)), \qquad \sqrt{|a(\phi)|}\nabla\phi \in L^\infty(0, T; L^2(\Omega)),$$
$$\partial_t\phi \in L^\infty(0, T_2; L^2(\Omega))$$

for some sufficiently small T_2. (In solving this, we can regularize the system to be strictly parabolic by the requirement $a > \nu > 0$.) Moreover, we have

$$\left\| \sqrt{|a(\phi)|}\nabla\phi \right\|_{L^\infty(0,T;L^2(\Omega))} + \|\phi\|_{L^\infty(0,T;L^\infty(\Omega))} + \|\partial_t\phi\|_{L^\infty(0,T;L^2(\Omega))}$$
$$\le N_2(N, T_2).$$

If T_2 is chosen sufficiently small, one can show that L is a bounded operator. By the compactness of the inverse to parabolic operator [20] and to the Navier–Stokes operator, one obtains the local existence of the weak solution of the system (46)–(48) with the parabolic regularization.

Then this solution satisfies the uniform estimates in Theorem 2, both in time T_2 (and the regularization); hence we can extend the time of the existence to any given value T. This gives the global existence of the weak solution in the space as indicated in Lemma 3.

In practice, we can solve the Navier–Stokes equations by using a Faedo–Galerkin finite-dimensional approximation. Then the existence of the approximate solution follows from Schauder's fixed point theorem. The convergence of the finite dimensional approximations will follow from the Lions–Aubin compactness results. We point out here that the energy estimate does not depend on the regularization. As $\nu \to 0$, we still have the limits in the corresponding energy spaces as in (35)–(37).

Of course, it might happen that the limiting function may not solve the original equation (46). On the other hand, the nature of the conservation law (46) precludes energy estimates if it is purely hyperbolic and needs completely different techniques when employing the viscosity method.

Remark 4. In case that a_1, b_1 are zero or small, the estimate (60) can be rewritten as

$$\frac{d}{dt}\int_\Omega\left(\frac{1}{2}\rho(\phi)|\mathbf{q}|^2+\frac{\varepsilon^2 a(\phi)}{2}|\nabla\phi|^2\right)d\mathbf{x}$$
$$+\left[\left(\mu_{\min}-e_2C(\delta)-C\phi_{\max}\right)\|\nabla\mathbf{q}\|^2_{L^2}-C(\|\phi\|_{H^1})\|\nabla\mathbf{q}\|_{L^2}\right]$$
$$+(\varepsilon^2-4\delta)\|\partial_t\phi\|_{L^2}\leq 0. \tag{65}$$

Hence, if $\mu_{\min}$ is large enough and ϕ_0 is small enough in the H^1 norm, then we have

$$\int_\Omega\left(\frac{1}{2}\rho(\phi)|\mathbf{q}|^2+\frac{\varepsilon^2 a(\phi)}{2}|\nabla\phi|^2\right)d\mathbf{x}\leq M. \tag{66}$$

Moreover, in this case, there holds

$$\int_0^\infty\int_\Omega(|\nabla\mathbf{q}|^2+|\partial_t\phi|^2)\,d\mathbf{x}\leq M,$$

and by Fubini's theorem, there exists a sequence $\{t_i\}_{i\in\mathbb{N}}$ such that

$$\int_\Omega\left(|\nabla\mathbf{q}(\mathbf{x},t_i)|^2+|\partial_t\phi(\mathbf{x},t_i)|^2\right)d\mathbf{x}\to 0 \qquad \text{as } t_i\to\infty; \tag{67}$$

i.e., $\nabla\mathbf{q}\to 0$ and $\partial_t\phi\to 0$ as $t_i\to\infty$, corresponding to the fact that the dynamical system approaches an equilibrium configuration. To derive a general decay rate, more detailed energy estimates will be necessary.

The system (46)–(48) can be obtained from the original problem (1)–(3) by linearization of the stress term with respect to ϕ. It retains both the mathematical and physical difficulties of the original problem, since the diffusion function a vanishes on an interval of positive solution values and the function f is nonlinear. Both phenomena give rise to the formation of discontinuities. However, we can see that the energy estimate does not depend on the regularization.

3.3. *Higher Order Regularization*

Equation (46) for ϕ does not make sense if $a(\cdot)$ is nonsmooth as is often the case in sedimentation models. Instead, we use $\varepsilon^2\Delta^2\phi$ as a regularization term. The function ϕ then no longer satisfies a maximum principle but still satisfies an L^∞ estimate. We then propose a different constitutive law instead of (5) by averaging the specific volumes: ρ and μ are defined by

$$\frac{1}{\rho(\phi)}=\frac{\phi}{\rho_{\mathrm{s}}}+\frac{1-\phi}{\rho_{\mathrm{f}}},\qquad \frac{1}{\mu(\phi)}=\frac{\phi}{\mu_{\mathrm{s}}}+\frac{1-\phi}{\mu_{\mathrm{f}}},$$

respectively, and will both be bounded away from zero.

In this section, we study the following higher order regularization in both the two- and three-dimensional cases:

$$\partial_t \phi + \mathbf{q} \cdot \nabla \phi + \nabla \cdot (f(\phi)\,\mathbf{k}) = \nabla \cdot (a(\phi)\nabla\phi) + \varepsilon^2 \Delta^2 \phi, \tag{68}$$

$$\nabla \cdot \mathbf{q} = 0, \tag{69}$$

$$\begin{aligned} \rho(\phi)(\partial_t \mathbf{q} + \mathbf{q} \cdot \nabla \mathbf{q}) - \nabla p_{\mathrm{e}} - \nabla \cdot (\mu(\phi)\mathbf{D}) \\ = \mathbf{a}_2 \phi + b_2 \nabla\phi + \mathbf{c}_2 \Delta\phi + e_2 \partial_t \nabla\phi + \varepsilon^2 \Delta\phi\nabla\phi. \end{aligned} \tag{70}$$

The last term

$$\varepsilon^2 \Delta\phi\nabla\phi = \varepsilon^2 \nabla \cdot (\nabla\phi \otimes \nabla\phi) - \varepsilon^2 \nabla\left(\frac{|\nabla\phi|^2}{2}\right)$$

is the (conservative) force from the (elastic) energy associated with the uneven distribution of ϕ. This term can be derived from the variational procedure. It also appears in several other complex fluid situations [13, 21]. Here we can directly use the batch function of the original system. Since the flux term $\nabla \cdot (f(\phi)\,\mathbf{k})$ now has not been linearized in Eq. (68), the terms b_1 and c_1 from Eq. (46) do not appear here. For the system with higher order regularization (68)–(70), we impose the boundary conditions

$$\mathbf{q}(\mathbf{x}, t) = 0, \quad \phi(\mathbf{x}, t) = \phi_0(\mathbf{x}), \quad \Delta\phi(\mathbf{x}, t) = 0 \qquad \text{for all } \mathbf{x} \in \partial\Omega, \quad t > 0, \tag{71}$$

or alternatively,

$$\mathbf{q}(\mathbf{x}, t) = 0, \quad \partial_{\mathbf{n}}\phi(\mathbf{x}, t) = 0, \quad \Delta\phi(\mathbf{x}, t) = 0 \qquad \text{for all } \mathbf{x} \in \partial\Omega, \quad t > 0, \tag{72}$$

and the initial condition

$$\mathbf{q}(x, 0) = \mathbf{q}_0(\mathbf{x}), \quad \phi(\mathbf{x}, 0) = \phi_0(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \Omega. \tag{73}$$

We point out again that ϕ does not satisfy the maximum principle in the sense of $\phi \in [0, \phi_{\max}]$ a.e. Rather, only an L^∞ estimate is valid.

To derive the energy estimates, we first consider the case $e_2 \equiv 0$; i.e., $\mathrm{D}_t\phi$ is not assumed to appear in the right-hand part of Eq. (70). Using integration by parts and Cauchy's inequality, we have the following energy estimate, noticing the cancellation of the last term in (70) and the convection term in (68):

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\rho(\phi)|\mathbf{q}|^2 + \frac{\varepsilon^2}{2}|\nabla\phi|^2\right) d\mathbf{x} \\ &\quad = -\int_\Omega \left(\mu(\phi)|\nabla\mathbf{q}|^2 + \varepsilon^2|\nabla\Delta\phi|^2\right) d\mathbf{x} + \int_\Omega \big((\mathbf{a}_2\phi \cdot \mathbf{q} + b_2 \nabla\phi\mathbf{q} \end{aligned}$$

$$+\mathbf{c}_2\cdot\Delta\phi\mathbf{q})\mathbf{q}+f(\phi)(\mathbf{k}\cdot\nabla)\Delta\phi+a(\phi)\nabla\phi\cdot\nabla\Delta\phi\big)\,d\mathbf{x}$$

$$\leq-\int_\Omega\left(\frac{\mu(\phi)}{2}|\nabla\mathbf{q}|^2+\frac{\varepsilon^2}{2}|\nabla\Delta\phi|^2\right)d\mathbf{x}$$

$$+C(\varepsilon)\int_\Omega(|\mathbf{q}|^2+|\nabla\phi|^2+|f(\phi)|)\,d\mathbf{x}. \tag{74}$$

Here we no longer need the assumption that $a(\phi)$ is smooth in ϕ, as we did in the last section. Neither do we need any condition on the batch flux density function $f(\phi)$. We only need the boundedness of the term $a(\phi)$. Estimate (74) then provides us with the following global estimate:

THEOREM 3. *Assume that $e_2\equiv 0$ in Eq.* (70), *and that $(\phi,\mathbf{q})$ is a solution of the system* (68)–(70). *Then to given T with $0<T<\infty$ there exists a constant $M(T,\varepsilon)$ depending on the coefficients and the initial data such that*

$$\int_\Omega\left(\frac{1}{2}|\mathbf{q}|^2+\frac{\varepsilon^2}{2}|\nabla\phi|^2\right)d\mathbf{x}\leq M(T,\varepsilon). \tag{75}$$

Moreover, this also yields the estimate

$$\int_0^T\int_\Omega\left(\mu(\phi)|\nabla\mathbf{q}|^2+\varepsilon^2|\nabla\Delta\phi|^2\right)d\mathbf{x}\leq 2M(T,\varepsilon). \tag{76}$$

As we see from Theorem 3, the bound $M(T,\varepsilon)$ depends on the given time T. As T is getting big, the constant can approach infinity. Nevertheless, this result suffices to prove the existence of the global weak solutions for the system (68)–(70), as sketched in Remark 3.

Remark 5. The regularity of the solution in Theorem 3 is due to the higher order regularizer $\varepsilon^2\Delta^2\phi$. If we let $\varepsilon\to 0$, the singularities of ϕ will develop. However, since the estimates are not uniform in ε, it is not clear what the limit will be like.

In case that e_2 is not zero, we need an additional estimate of the type

$$\left|\int_\Omega\Delta^2\phi\mathbf{q}\;d\mathbf{x}\right|=\left|\int_\Omega\nabla\Delta\phi\cdot\nabla\mathbf{q}\;d\mathbf{x}\right|\leq\delta\|\nabla\Delta\phi\|_{L^2}^2+C\|\nabla\mathbf{q}\|_{L^2}^2. \tag{77}$$

Combining this with the estimates in (74), we arrive at the following conclusion.

THEOREM 4. *Assume that $(\phi,\mathbf{q})$ is a solution of the system* (68)–(70). *There exists a constant $K(\varepsilon)$ depending on the coefficients and the initial data, such that if*

$$\mu_{\min}>K(\varepsilon), \tag{78}$$

then the estimates in Theorem 2 *hold again.*

Again, this theorem guarantees the existence of the global weak solutions according to Remark 3.

4. CONCLUSIONS

We here study three regularized problems, (13–(15), (46)–(48) and (68)–(70) which do not completely coincide with the sedimentation–consolidation system (1)–(3) introduced in [1, 7]. However, the present work illustrates some fundamental properties of the systems from the energetical point of view. The energy estimates of all the systems discussed in this paper show that these systems are dissipative and naturally agree with the second law of thermodynamics. It also show that the dynamic system will approach certain hydrodynamic equilibrium as time tends to infinity.

All the analysis in this paper is carried out in spatially two- or three-dimensional domains. In order to study the properties of the singularities, such as discontinuities and shocks as the regularization parameter ε vanishes, our future projects will be to combine these estimates with other more refined treatments of higher dimensional strongly degenerating parabolic–hyperbolic scalar equations. These will employ the techniques of entropy solutions by Kružkov [17] and their extensions to second order equations [9, 16] (which were also applied to some simplified sedimentation–consolidation systems in [2, 5]), as well as the work by Di Perna and Lions [11] for scalar transport equations.

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